Exercise 5

Use the *series solution method* to solve the Volterra integral equations of the first kind:

$$-1 - x + \frac{1}{6}x^3 + e^x = \int_0^x (x-t)u(t)\,dt$$

Solution

We seek a series solution for u:

$$u(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots .$$

Substitute this and the Taylor series expansion of e^x,

$$e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5 + \cdots ,$$

into the integral equation.

$$\begin{aligned} -1 - x + \frac{1}{6}x^3 + \left(1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5 + \cdots\right) \\ = \int_0^x (x-t)(a_0 + a_1 t + a_2 t^2 + a_3 t^3 + \cdots)\,dt \end{aligned}$$

Simplify the left side and evaluate the integral on the right side.

$$\frac{1}{2}x^2 + \frac{1}{3}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5 + \cdots = \frac{a_0}{2}x^2 + \frac{a_1}{6}x^3 + \frac{a_2}{12}x^4 + \frac{a_3}{20}x^5 + \cdots$$

Match the coefficients of the respective powers of x to determine a_i.

$$\begin{aligned} \frac{a_0}{2} &= \frac{1}{2} & &\rightarrow & a_0 &= 1 \\ \frac{a_1}{6} &= \frac{1}{3} & &\rightarrow & a_1 &= 2 \\ \frac{a_2}{12} &= \frac{1}{24} & &\rightarrow & a_2 &= \frac{1}{2} \\ \frac{a_3}{20} &= \frac{1}{120} & &\rightarrow & a_3 &= \frac{1}{6} \\ &\vdots & & & &\vdots \end{aligned}$$

So then

$$\begin{aligned} u(x) &= 1 + 2x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \cdots \\ &= x + \left(1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \cdots\right) \\ &= x + e^x . \end{aligned}$$